International Journal of Theoretical & Applied Sciences, Special Issue-NCRTAST 8(1): 175-179(2016)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

A Common Fixed Point Theorem in Complete Fuzzy 3 - Metric Spaces

V. K. Agrawal*, Ajay Soni[**] and Anupama Gupta**

[*]Department of Mathematics, Govt. H. S. College, H. bad, India,
[**]Research Scholar Barkatullah University, Bhopal, India,
[***]Department of Mathematics, BUIT, Bhopal, INDIA

(Corresponding author: Ajay Soni)
(Received 11 April, 2016 Accepted 20 May, 2016)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: Mishra *et.al* (2008) proved a common fixed point theorem in fuzzy metric space by introducing reciprocal continuity. In this paper we have extended and generalize the above result for fuzzy 3- metric spaces.

Keywords: Fuzzy 2–metric space, Fuzzy 3–metric space, common fixed point theorem.

I. INTRODUCTION

Fixed point theorems in fuzzy metric spaces satisfying some contractive condition are a central area of research now a day. The concept of fuzzy sets was introduced by Zadeh [10] in 1965. After this fuzzy set theory was further developed and a series of research were done by several Mathematicians. Kramosil and Michlek [5] introduced the concept of fuzzy metric space in 1975 and fixed point theorems for fuzzy metric space was first obtained by Helpern [4] in 1981. Later in 1994, George and Veeramani [3] modified the notion of fuzzy metric space with the help of t-norm. Some fixed point theorem in metric space are generalized to fuzzy metric space by several authors. There are various ways to define a fuzzy metric space, here we adopt the notion that, the distance between objects is fuzzy, the objects themselves may be fuzzy or not. Dersanambika and Aswathy and Gahler [1], [2] investigated the properties of 2-metric space in his papers, and many authors investigated contraction mappings in 2-metric spaces. Succeeding this, the notion of 3-metric space was also introduced. We know that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in the Euclidian space, whereas the 3-metric space was suggested by the volume function. The idea of fuzzy 2-metric space and fuzzy 3-metric space were used by Sharma [8] and obtained some fruitful results. Motivated by Sharma [8], we prove some common fixed point theorem in fuzzy 2-metric space and fuzzy 3-metric space by employing the notion of reciprocal continuity, of which we can widen the scope of many interesting fixed point theorems in fuzzy metric space.

II. PRELIMINARY NOTES

Definition 2.1. A triangular norm * is a binary operation on the unit interval [0, 1] such that for all a, b, c, d ∈ [0, 1] the following conditions are satisfied:
1. a *1 = a;
2. a * b = b * a;
3. a * b ≤ c *d whenever a ≤ c and b ≤ d
4. a * (b * c) = (a * b) * c.

Definition 2.2. The 3-tuple (X, M, *) is called a fuzzy metric space, if X is an arbitrary set, * is a continuous t - norm and M is a fuzzy set in $X^2 \times [0, \infty]$ satisfying the following conditions: for all x, y, z ∈ X and s, t > 0
C'- 1 M(x, y, 0) = 0
C'- 2 M(x, y, t) = 1, for all t > 0, if and only if x = y
C'- 3 M(x, y, t) = M(y, x, t)
C'- 4 M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s)
C'- 5 M(x, y, .) : [0,∞) → [0, 1] is left continuous,
C'- 6 $\lim_{t \to \infty} M(x, y, t) = 1$

Example 2.3. Let (X, d) be a metric space. Define a * b = ab (or a * b = min {a, b}) and for all x, y ∈ X and t > 0,
M(x, y, t) = $\frac{t}{t+d(x,y)}$.
Then (X, M, *) is a fuzzy metric space and this metric d is the standard fuzzy metric.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to converge to x in X if and only if M (x_n, x, t) = 1 for each t > 0.

Definition 2.5. Let (X, M, *) be a fuzzy metric space A sequence $\{x_n\}$ in X is called Cauchy sequence if and only if M (x_{n+p}, x_n, t) = 1 for each p > 0, t > 0.

Definition 2.6. A fuzzy metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.7. A pair (f, g) of self maps of a fuzzy metric space (X, M, *) is said to be reciprocal continuous if
$\lim_{n\to\infty} fgx_n$ = fx and $\lim_{n\to\infty} gfx_n$ = gx
Then there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n$
$\lim_{n\to\infty} gx_n$ = x for some x ∈ X

Definition 2.8. Two self maps A and B of a fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, that is Ax = Bx Implies ABx = BAx.

Definition 2.9. A pair (A, S) of self maps of a fuzzy metric space (X, M, *) is said to be semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence $\{x_n\}$ in X such
that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ = x for some x ∈ X.

Definition 2.10. A binary operation * : [0, 1] × [0, 1] × [0, 1] → [0, 1] is called a continuous t - norm if ([0, 1]), *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \le a_2 * b_2 * c_2$ whenever $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ are in [0, 1].

Definition 2.11. The 3-tuple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^3 \times [0, \infty]$ satisfying the following conditions: for all x, y, z, u ∈ X and $t_1, t_2, t_3 > 0$.

C”-1 M(x, y, z, 0) = 0,

C”- 2 M(x, y, z, t) = 1, t > 0 and when at least two of the three points are equal,

C”- 3 M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) (Symmetry about three variables)

C”- 4 M(x, y, z, $t_1 + t_2 + t_3$) ≥ M(x, y, u, t_1) * M (x, u, z, t_2) * M (u, y, z, t_3)

(This corresponds to tetrahedron inequality in 2-metric space)

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

C”- 5 M(x, y, z, .): [0, ∞) → [0, 1] is left continuous.

Definition 2.12. A sequence $\{x_n\}$ in a fuzzy 2 - metric space (X, M, *) is said to converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all a ∈ X and t > 0.

Definition 2.13. Let (X, M, *) be a fuzzy 2-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if and only if
$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all a ∈ X and p > 0, t > 0 .

Definition 2.14. A fuzzy 2 - metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent in X.

Definition 2.15. A binary operation *: [0, 1] × [0, 1] × [0, 1] × [0, 1] → [0, 1] is called a continuous t- norm if ([0, 1]), *) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \le a_2 * b_2 * c_2 * d_2$ whenever $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2, d_1 \le d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are in [0, 1].

Definition 2.16. The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t - norm and M is a fuzzy set in $X^4 \times [0, \infty]$ satisfying the following conditions: for all x, y, z, w, u ∈ X and $t_1, t_2, t_3, t_4 > 0$.

C-1 M(x, y, z, w, 0) = 0,

C- 2 M(x, y, z, w, t) = 1, for all t > 0, (Only when the three simplex < x, y, z, w > degenerate)

C- 3 M (x, y, z, w, t) = M (x, w, z, y, t) = M (y, z, w, x, t) = M (z, w, x, y, t) =…...

C- 4 M (x, y, z, w, $t_1 + t_2 + t_3 + t_4$) ≥ M (x, y, z, u, t_1) * M (x, y, u, w, t_2) * M (x, u, z, w, t_3) * M (u, y, z, w, t_4)

C- 5 M (x, y, z, w, .) : [0, ∞) → [0, 1] is left continuous.

Definition 2.17. A sequence $\{x_n\}$ in a fuzzy 3-metric space (X, M, *) is said to converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$ for all a, b ∈ X and t > 0.

Definition 2.18. Let (X, M, *) be a fuzzy 3-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1$
for all a, b ∈ X , p > 0 , and t > 0.

Definition 2.19. A fuzzy 3-metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent in X.

III. MAIN RESULTS

Theorem 3.1: Let (X, M, *) be a complete fuzzy metric space with continuous t-norm. Let A, B, S, C, R and T be self maps on a complete fuzzy metric space satisfying q ∈ (0, 1) for all x, y ∈ X and t > 0.

$$M(Ax, By, Cz, u, t) \ge \left[\alpha\, M(Rx, Sy, Tz, u, t) + \beta \min\begin{Bmatrix} M(Rx, Sy, Tz, u, t) \\ M(Ry, Sz, Tx, u, t) \\ M(Rz, Sx, Ty, u, t) \end{Bmatrix}\right]$$

Where $\alpha \ge 1$ then A, B, Sand T have a common fixed point.

Proof: Suppose $x_0 \in X$ be an arbitrary point. Then there exists $x_1, x_2, x_3 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$ and $Cx_2 = Rx_3$. Thus we can form sequences $\{z_n\}$, $\{y_n\}$, $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, $y_{2n+3} = Cx_{2n+2} = Rx_{2n+3}$, for n = 0, 1, ...

$$M(Ax, By, Cz, u, t) \geq \left[\alpha\, M(Rx, Sy, Tz, u, t) + \beta \min \begin{Bmatrix} M(Rx, Sy, Tz, u, t) \\ M(Ry, Sz, Tx, u, t) \\ M(Rz, Sx, Ty, u, t) \end{Bmatrix}\right]$$

$$M(y_{2n+1}, y_{2n+2}, y_{2n+3}, u, t) = M(Ax_{2n}, Bx_{2n+1}, Cx_{2n+2}, u, t)$$

$$\geq \left[\begin{matrix} \alpha\, M(Rx_{2n}, Sx_{2n+1}, Tx_{2n+2}, u, t) \\ +\beta \min \begin{Bmatrix} M(Rx_{2n}, Sx_{2n+1}, Tx_{2n+2}, u, t) \\ M(Rx_{2n+1}, Sx_{2n+2}, Tx_{2n}, u, t) \\ M(Rx_{2n+2}, Sx_{2n}, Tx_{2n+1}y, u, t) \end{Bmatrix} \end{matrix}\right]$$

$$\geq \left[\begin{matrix} \alpha\, M(y_{2n}, y_{2n+1}, y_{2n+2}, u, t) \\ +\beta \min \begin{Bmatrix} M(y_{2n}, y_{2n+1}, y_{2n+2}, u, t) \\ M(y_{2n+1}, y_{2n+2}, y_{2n}, u, t) \\ M(y_{2n+2}, y_{2n}, y_{2n+1}, u, t) \end{Bmatrix} \end{matrix}\right]$$

$$M(y_{2n+1}, y_{2n+2}, y_{2n+3}, u, t) \geq [(\alpha + \beta)\, M(y_{2n}, y_{2n+1}, y_{2n+2}, u, t)]$$

If $(\alpha + \beta) \geq 1$, then $M(y_{2n+1}, y_{2n+2}, y_{2n+3}, u, t) \geq M(y_{2n}, y_{2n+1}, y_{2n+2}, u, t)$

Similarly $M(y_{2n+3}, y_{2n+4}, y_{2n+5}, u, t) > (M(y_{2n+2}, y_{2n+3}, y_{2n+4}, u, t))$.

More generally, $M(y_{n+1}, y_n, y_{n-1}, u, t) > (M(y_n, y_{n-1}, y_{n-2}, u, t))$

Therefore $\{M(y_{n+1}, y_n, y_{n-1}, u, t)\}$ is an increasing sequence of positive real numbers in [0, 1] and tends to limit l ≤ 1. We claim that l = 1. If l < 1 then

$M(y_{n+1}, y_n, y_{n-1}, u, t) > (M(y_n, y_{n-1}, y_{n-2}, u, t))$.

On taking $n\to\infty$ we get,

$\lim_{n\to\infty} M(y_{n+1}, y_n, y_{n-1}, u, t) > \lim_{n\to\infty} M(y_n, y_{n-1}, y_{n-2}, u, t)$.

Now for any positive integer p.

$$\begin{aligned} M(y_n, y_{n+1}, y_{n+p}, u, t) \geq\ & M\left(y_n, y_{n+1}, y_{n+2}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \\ & * M\left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \\ & * M\left(y_{n+2}, y_{n+3}, y_{n+4}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \\ & * \ldots * M\left(y_{n+p-3}, y_{n+p-2}, y_{n+p-1}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \\ & * M\left(y_{n+p-2}, y_{n+p-1}, y_{n+p}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \\ & = M\left(y_n, y_{n+1}, y_{n+2}, z, \tfrac{t}{3(p-1)+1}\right) \\ & * M\left(y_{n+1}, y_{n+2}, y_{n+3}, z, \tfrac{t}{3(p-1)+1}\right) \\ & * M\left(y_{n+2}, y_{n+3}, y_{n+4}, z, \tfrac{t}{3(p-1)+1}\right) \\ & * \ldots * M\left(y_{n+p-3}, y_{n+p-2}, y_{n+p-1}, z, \tfrac{t}{3(p-1)+1}\right) \\ & * M\left(y_{n+p-2}, y_{n+p-1}, y_{n+p}, z, \tfrac{t}{3(p-1)+1}\right) \end{aligned}$$

Taking limits as $n\to\infty$ we have

$$\begin{aligned} \lim_{n\to\infty} M(y_n, y_{n+1}, y_{n+p}, u, t) \geq\ & \lim_{n\to\infty} M\left(y_n, y_{n+1}, y_{n+2}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \\ & * \lim_{n\to\infty} M\left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+p}, \tfrac{t}{3(p-1)+1}\right) \end{aligned}$$

$$* \lim_{n\to\infty} M(y_{n+2}, y_{n+3}, y_{n+4}, y_{n+p}, \frac{t}{3(p-1)+1})$$
$$*\ldots..* \lim_{n\to\infty} M(y_{n+p-3}, y_{n+p-2}, y_{n+p-1}, y_{n+p}, \frac{t}{3(p-1)+1})$$
$$\geq \lim_{n\to\infty} M(y_n, y_{n+1}, y_{n+2}, z, \frac{t}{3(p-1)+1})$$
$$* \lim_{n\to\infty} M(y_{n+1}, y_{n+2}, y_{n+3}, z, \frac{t}{3(p-1)+1})$$
$$* \lim_{n\to\infty} M(y_{n+2}, y_{n+3}, y_{n+4}, z, \frac{t}{3(p-1)+1})$$
$$*\ldots....* \lim_{n\to\infty} M(y_{n+p-3}, y_{n+p-2}, y_{n+p-1}, z, \frac{t}{3(p-1)+1})$$

that is

$\lim_{n\to\infty} M\,(y_n, y_{n+1}, y_{n+p}, u, t) \geq 1 * 1 * 1 * 1 * \ldots * 1 = 1$

Which means $\{y_n\}$ is a Cauchy sequence in X. Since X is complete $y_n \to$w in X. That is $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$,$\{Sx_{2n+2}\}$, $\{Cx_{2n+2}\}$, $\{Rx_{2n+3}\}$ also converges to w in X.

That is $\lim_{n\to\infty} Rx_{2n} \to w$, $\lim_{n\to\infty} Ax_{2n} \to w$. Since (A, R) is semi compatible, $\lim_{n\to\infty} ARx_{2n} \to R\,w$. Also (*A*,*R*) is reciprocal continuous also, therefore, $\lim_{n\to\infty} RAx_{2n} \to Aw$. Combining this process we get *Aw* = *Rw*.

Now to prove that *Aw* = w, for if we consider that $Aw \neq$ w. Then by the contractive condition,

$$M\,(Aw, Bx_{2n+1}, Cx_{2n+2}, u, t) \geq \begin{bmatrix} \alpha\, M\,(Rw, Sx_{2n+1}, Tx_{2n+2}, u, t) \\ +\beta \min \begin{Bmatrix} M\,(Rw,\ Sx_{2n+1}, Tx_{2n+2}, u, t) \\ M\,(Rx_{2n+1}, Sx_{2n+2}, Tw, u, t) \\ M\,(Rx_{2n+2}, Sw, Tx_{2n+1}, u, t) \end{Bmatrix} \end{bmatrix}$$

$$M\,(Aw, w, w, u, t) \geq \begin{bmatrix} \alpha\, M\,(Rw, w, w, u, t) \\ +\beta \min \begin{Bmatrix} M\,(Rw,\ w, w, u, t) \\ M\,(w, w, w, u, t) \\ M\,(w, w, w, u, t) \end{Bmatrix} \end{bmatrix}$$

$$M\,(Aw, w, w, u, t) \geq (\alpha + \beta)\, M\,(Rw, w, w, u, t)$$

$$> M(Aw, w, w, u, t) \text{ for } (\alpha + \beta) \geq 1$$

Therefore Aw = w = Rw.

Since (B, S) and (C, T) are weakly compatible and reciprocally continuous, as above we get Bw = w = Sw and Cw = w = Tw. Therefore A, B, C, R, S and T has a common fixed point.

REFERENCES

[1]. Dersanambika K.S. & Aswathy M.R: Fixed Point Theorems in Fuzzy Metric Spaces, *J. Contemp. Math.*. **6**, (2011), 1079 – 1089

[2]. Gahler, S.: Linear 2-normierte Raume, *Math. Nachr.*, **28**(1964), 143.

[3]. Gahler, S.: ¨U ber 2-Banach Raume, *Math. Nachr.* **42**(1969), 335347.

[4]. George, A., Veeramani, P.: On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64 (1994), 395 – 399.

[5]. Heipern, S.: Fuzzy mappings and fixed point theorems, *J. Math. Anal. Appl.*, **83**(1981), 566-569.

[6]. Kramosil, O. & Michalek, J.: Fuzzy metric and statistical metric spaces, *Kybernelika*, **11** (1975), 326-334.

[7]. Singh, B. & Jain, S.: Semi-compatible and fixed point theorems in fuzzy metric spaces, *Chungcheong Math. Soc*, **18**(2005), 1-22.
[8]. Sharma, P. L., Sharma, B. K., Iseki, K.: Contractive type mapping on 2-metric space, *Math. Japonica*, **21**(1976), 6770.
[9]. Sushil S.: On fuzzy metric space, *Southeast Asian Bulletin of Mathematics*, **26**, 2002, 133-145.
[10]. Mishra, U., Ranavide, A. S. and Dhananjay, G.: Some fixed point theorems in fuzzy metric space, *Tamkang Journal of Mathematics*, **39**(4)(2008), 309-316.
[11]. Zadeh, L. A.: Fuzzy sets, *Inform and Control*, **8**(1965), 338-353.